Internal clock formulation of quantum mechanics

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Time in the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle, \quad |\Psi\rangle \in Dom(\hat{H}) \subseteq \mathcal{H}$$

1. In the Newtonian mechanics, the parameter 't' labels the flow of the absolute (i.e. fixed and external) time. The absolute time is not measurable. In practice, one has to rely on a dynamical, internal degree of freedom (an internal component of the global system), let us call it **internal clock**, which well imitates the flow of the absolute time.
2. Therefore, in **quantum** mechanics, there exists the problem of making a choice of a **classical** degree of freedom that evolves at pace with the absolute time.
3. The problem gets even more serious when one works with more fundamental theories like General Relativity (or, more broadly, with **Hamiltonian constraint systems**). There is no external and fixed time and the spacetime is a dynamical entity. In particular, **time is internal and dynamical, and its choice is ambiguous**.

Usual approach to the problem

$$i\frac{\partial}{\partial\phi}|\Psi\rangle = \hat{H}_\phi|\Psi\rangle, \quad |\Psi\rangle \in Dom(\hat{H}_\phi) \subseteq \mathcal{H}$$

1. The parameter ϕ is an ad-hoc, usually chosen for simplicity, internal degree of freedom in terms of which the dynamics of the remaining degrees of freedom is expressed.
2. The rarely addressed issues are: (1) Is there any **physical justification** of the choice of ϕ? (2) If not, then how to implement the **freedom of the choice of clock** into the formalism of quantum mechanics?
3. Maybe interesting but what for...? Theories of the origin of structure in the universe like inflation are based on **quantization of gravitational degrees of freedom**. The distribution of matter is a random variable as the outcome of the wave-function collapse.

Basic tool: pseudocanonical transformations

1. We shall consider clock transformations $\phi \mapsto \bar{\phi} = \bar{\phi}(\phi, q, p)$
2. We extend the H - J theory of canonical transformations $(q, p, \phi) \mapsto (\bar{q}, \bar{p})$:

$$\omega_{\mathcal{C}} = \mathrm{d}q\mathrm{d}p - \mathrm{d}\phi\mathrm{d}H = \mathrm{d}\bar{q}\mathrm{d}\bar{p} - \mathrm{d}\phi\mathrm{d}\bar{H}$$

   (where the symplectic form is defined as $\omega_{\mathcal{C}}|_{\phi}$), to the theory of **pseudocanonical transformations** $(q, p, \phi) \mapsto (\bar{q}, \bar{p}, \bar{\phi})$:

$$\omega_{\mathcal{C}} = \mathrm{d}q\mathrm{d}p - \mathrm{d}\phi\mathrm{d}H = \mathrm{d}\bar{q}\mathrm{d}\bar{p} - \mathrm{d}\bar{\phi}\mathrm{d}\bar{H}$$

3. We need to implement these transformations at the quantum level. It is sufficient to study a section σ given by of $2n+1$ algebraic equations:

$$\boxed{\bar{t} = \bar{t}(t, q, p),\ \ C_I(t, q, p) = C_I(\bar{t}, \bar{q}, \bar{p}),\ \ I = 1, \ldots, 2n}$$

   where C_I is a conserved quantity.

Properties of the extended quantum formalism

1. A physical state is given by a vector $|\Psi\rangle \in \mathcal{H}$
2. Non-dynamical content of $|\Psi\rangle \in \mathcal{H}$ is determined by self-adjoint Dirac observables C and is independent of the choice of internal clock

$$\Psi(c) = \langle \phi_c | \Psi \rangle \in L^2(sp(\hat{C}), \mathrm{d}c), \quad \hat{C}|\phi_c\rangle = c|\phi_c\rangle$$

3. The Hamiltonian is a Dirac observable, the quantum evolution (trajectories in $\mathcal{H}$) is independent of the choice of internal clock

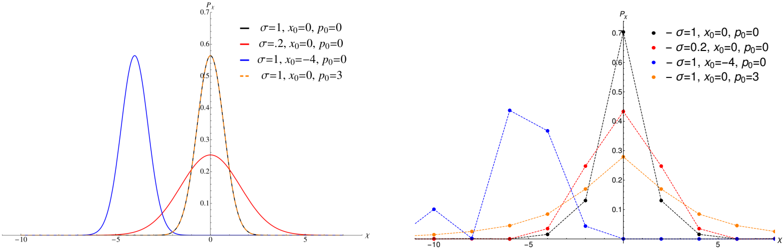
$$\mathbb{R} \ni \phi \mapsto |\Psi(\phi)\rangle \in \mathcal{H}, \quad \{\phi\} \in \mathcal{T}$$

4. Dynamical content of $|\Psi\rangle \in \mathcal{H}$ is provided by means of self-adjoint dynamical observables $O \mapsto \hat{O}_\phi$ and crucially depends on the choice of internal clock,

$$\Psi_\phi(o) = \langle \phi_{o,\phi} | \Psi \rangle \in L^2(sp(\hat{O}_\phi), \mathrm{d}o), \quad \hat{O}_\phi|\phi_{o,\phi}\rangle = o|\phi_{o,\phi}\rangle$$

5. Ordinary QM is included as a special case

Example: a free particle on the line, $\bar{\phi} = \phi + qp$



Probability distribution $P_\chi = |\langle \chi|\psi\rangle|^2$ of position eigenvalues for a fixed (gaussian) state $|\psi\rangle$ in the initial clock ϕ (on the left) and in the new clock $\bar{\phi} = \phi + qp$ (on the right). The position spectrum in $\bar{\phi}$ is discrete and marked with dots.

How to obtain the ordinary QM?

Divide the total system into a product of **internal system** and **internal observer**: $(q_s, p_s) \times (q_o, p_o) \in \mathbb{R}^4, \quad \phi \in \mathbb{R}$.

$$\omega_{TOTAL} = \omega_s + \omega_o, \quad H_i = \frac{p_i^2}{2}, \quad i = s, o$$

and let the clock transformation involve **internal observer** only:

$$\phi \mapsto \bar{\phi} = \phi + D(q_o, p_o).$$

Internal observer: $\omega_o|_{\bar{\phi}} \neq \omega_o|_{\phi}$, (*pseudocanonical*),

Internal system: $\omega_s|_{\bar{\phi}} = \omega_s|_{\phi + D(q_o(\phi), p_o(\phi))}$, (*canonical*).

The ϕ- and $\bar{\phi}$-frames of quantized **internal system** are related by a unitarity $U = e^{-\frac{iD}{2}\hat{P}^2}$:

Clock ϕ	Clock $\bar{\phi} = \phi + D(\phi)$
$p_s \mapsto \hat{P}$ $q_s \mapsto \hat{Q}$ $\|\Psi\rangle \mapsto \Psi(q) = \langle q\|\Psi\rangle$ $i\partial_{\phi}\psi(q) = \hat{H}\psi(q)$	$p_s \mapsto \hat{P}$ $q_s \mapsto \hat{Q} - D(\phi)\hat{P}$ $\|\Psi\rangle \mapsto \varphi(q) = \langle q\|U^{\dagger}\|\Psi\rangle$ $i\partial_{\bar{\phi}}\varphi(q) = \hat{H}\varphi(q)$