Classical and quantum chaos in gravity

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Belinskii-Khalatnikov-Lifshitz (BKL) conjecture

- Einstein's theory of gravity (general relativity) is known to suffer from gravitational singularities (incomplete geodesics and diverging invariants)
- BKL conjecture states: general relativity implies existence of generic general solution that is singular
  - corresponds to non-zero measure subset of all initial data
  - is stable against perturbation of initial data
  - depends on proper number of arbitrary functions defined on space part of spacetime

V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982)

- Remarks:
  - Evolution of spacetime that leads to BKL singularity is called BKL scenario.
  - BKL scenario presents a very complicated dynamics so that to work with it one needs to use models.

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Dynamics underlying BKL scenario

The massive model of BKL scenario

Derived by V. Belinski, I. Khalatnikov, and M. Ryan in 1971; E. Czuchry and W. P., Phys. Rev. D **87**, 084021 (2013)

$$\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}, \tag{1}$$

where $a = a(t),\ b = b(t),\ c = c(t)$ are effective directional scale factors, and t is a monotonic function of proper time.

The solutions to (1) must satisfy the constraint

$$\frac{d \ln a}{dt}\frac{d \ln b}{dt} + \frac{d \ln a}{dt}\frac{d \ln c}{dt} + \frac{d \ln b}{dt}\frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{2}$$

Eqs (1)-(2) present essence of dynamics underlying BKL scenario.

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Solution to the model of BKL scenario

We have found exact solution to the dynamics (1)–(2):

P. Goldstein and W.P., Eur. Phys. J. C (2022) 82: 216

$$\tilde{a}(t) = \frac{3}{t - t_0}, \quad \tilde{b}(t) = \frac{30}{(t - t_0)^3}, \quad \tilde{c}(t) = \frac{120}{(t - t_0)^5}, \tag{3}$$

where $t > t_0$ and where t_0 is an arbitrary real number.

The solution (3) is unstable against small perturbation:

$$a(t) = \tilde{a}(t) + \epsilon\alpha(t), \tag{4a}$$

$$b(t) = \tilde{b}(t) + \epsilon\beta(t), \tag{4b}$$

$$c(t) = \tilde{c}(t) + \epsilon\gamma(t), \tag{4c}$$

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Solution to BKL scenario (cont)

Inserting (4) into (1)–(2) leads, in the first order in ϵ, to the following solution of the resulting equations:

$$\alpha(t) = \exp(-\theta/2)\big[K_1\cos(\omega_1\theta+\varphi_1)+K_2\cos(\omega_2\theta+\varphi_2)\big]+K_3\exp(-2\theta), \tag{5a}$$

$$\beta(t) = \exp(-5\theta/2)\big[(4+6\sqrt{6})K_1\cos(\omega_1\theta+\varphi_1) \tag{5b}$$

$$+ (4-6\sqrt{6})K_2\cos(\omega_2\theta+\varphi_2)\big] + 30K_3\exp(-4\theta), \tag{5c}$$

$$\gamma(t) = -4\exp(-9\theta/2)\big[(26+9\sqrt{6})K_1\cos(\omega_1\theta+\varphi_1) \tag{5d}$$

$$+ (26-9\sqrt{6})K_2\cos(\omega_2\theta+\varphi_2)\big] + 200K_3\exp(-6\theta)\,, \tag{5e}$$

where $\theta = \ln(t-t_0)$. The two frequencies read

$$\omega_1 = \frac{1}{2}\sqrt{95-24\sqrt{6}}, \qquad \omega_2 = \frac{1}{2}\sqrt{95+24\sqrt{6}}\,, \tag{6}$$

where K_1, K_2, K_3, φ_1, and φ_2 are constants.

Chaotic phase of BKL scenario

- The manifold $\mathcal{M}$ defined by $\{K_1, K_2, K_3, \varphi_1, \varphi_2\}$ is a submanifold of $\mathbb{R}^5$. Thus, (5) presents generic solution as the measure of $\mathcal{M}$ is nonzero.
- The relative perturbations $\alpha/a, \beta/b$, and γ/c grow as $\exp(\frac{1}{2}\theta)$.
  - The multiplier $1/2$ plays the role of a Lyapunov exponent, describing the rate of their divergences.
  - Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) becomes chaotic.
- Chaoticity results from strong nonlinearity of the dynamics and growing curvature of spacetime in evolution towards singularity.

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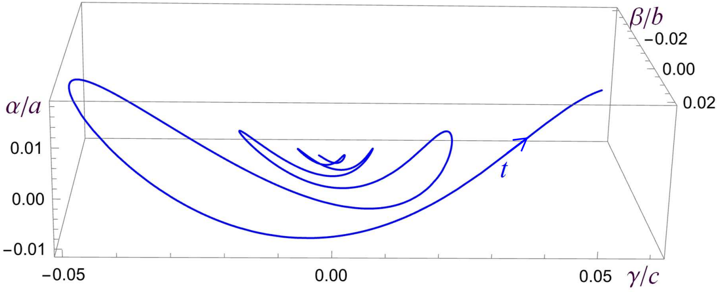
Chaotic phase of the BKL scenario (cont)



Parametric curve presenting growing instability of scale factors in evolution towards singularity.

Quantization of chaotic phase of the BKL scenario

BKL scenario can serve as sophisticated model of evolution of the Universe near cosmological singularity. It is highly interesting to see what happens to classical chaos at quantum level.

- We quantize BKL scenario by making use of integral quantization method (which we develop in our Department).
- We do not quantize Hamilton's dynamics, but the solution to the BKL scenario (presented earlier).
- We quantize both temporal and spatial variables to support general covariance of GR with respect to transformations of these variables.

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Quantization of the BKL scenario (cont)

Outline of calculations

For details, see: A. Góźdź, A. Pędrak, and W.P., arXiv:2204.11274 [gr-qc].

- We calculate variances of quantum observables corresponding to perturbed $\{a, b, c\}$ and unperturbed $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ solutions.
- Variances describe stochastic deviations (quantum smearing) from expectation values of quantum observables.

We have described quantum instabilities as follows

$$\kappa_k := \frac{var(\hat{\xi}_k; \Psi_{pert}) - var(\hat{\xi}_k; \Psi_{unpert})}{var(\hat{\xi}_k; \Psi_{unpert})}, \qquad k = a, b, c \tag{7}$$

where $\hat{\xi}_a := \hat{a},\ \hat{\xi}_b := \hat{b},\ \hat{\xi}_c := \hat{c}$.

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Stochastic aspects of quantum evolution

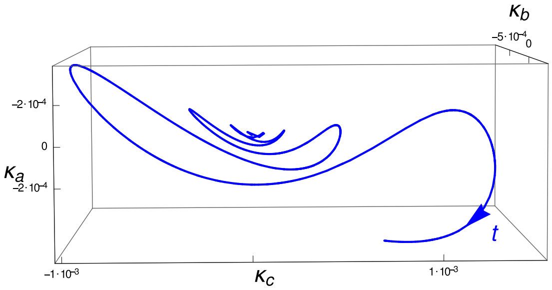

Figure: Parametric curve of relative quantum instabilities.

Conclusions

- Evolution of classical gravitational system towards generic singularity is chaotic.
  The corresponding quantum evolution is definitely stochastic.
- Nonlinearity of singular classical dynamics may create deterministic chaos.
  Non-vanishing variances of observables of the corresponding quantum dynamics may create stochastic chaos.

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Thank you!

Variance of quantum observable

Variance is a stochastic deviation from expectation value of quantum observable; it determines the value of smearing of quantum observable.
The variance is the average of the squared differences from the mean.
In the quantum state labelled by ψ, the variance is defined to be

$$var(\hat{A};\psi) := \langle(\hat{A} - \langle\hat{A};\psi\rangle)^2;\psi\rangle = \langle\hat{A}^2;\psi\rangle - \langle\hat{A};\psi\rangle^2\,, \tag{8}$$

where $\langle\hat{B};\psi\rangle := \langle\psi|\hat{B}|\psi\rangle$.
If $\hat{A}$ is a self-adjoint operator, we have the important statement:

$$\Big(var(\hat{A};\psi) = 0\Big) \Longleftrightarrow \Big(\hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R}\Big)\,, \tag{9}$$

i.e., the variance of the operator $\hat{A}$ equals zero, if and only if, the quantum system is in an eigenstate of the operator $\hat{A}$.